Penrith Selective High School
Mathematics Extension 2
Trial HSC 2019

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Calculators approved by NESA may be used
- No correction tape or white out to be used
- A reference sheet is provided with this paper
- Use black pen
- In Questions 11–16, show relevant mathematical reasoning and/ or calculations

	Complex Numbers	Graphing	Polynomials	Integration	Conics	Volumes	Harder Ext 1	Resisted Motion	Total
Mult. Choice									/10
Q11									/15
Q12									/15
Q13									/15
Q14									/15
Q15									/15
Q16									/15
Total	/11	/11	/7	/10	/9	/18	/19	/15	/100

Student Number:______________________________________

Teacher's Name:_______________________________________

Section I (10 marks)

Attempt Questions 1–10

Allow about 15 minutes for this section

Use the Blue multiple-choice answer sheet for Questions 1–10.

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1. Simplify $(5+2i)^2 - (3-2i)^2$

A. $4(4+i)$

B. $8(2+i)$

C. $16(1+2i)$

D. $16(1+i)$

2. Solve $x^4 + 2x^3 - 12x^2 + 14x - 5 = 0$, given that it has a triple root.

A. $x = 1, 1, 1, 5$

B. $x = -1, -1, -1, 5$

C. $x = -1, -1, -1, -5$

D. $x = 1, 1, 1, -5$

3. Evaluate $\int_0^1 \frac{x^2}{x+1}\, dx$

A. $\frac{-3 - ln4}{2}$

B. $\frac{-1 + ln4}{2}$

C. $\frac{-1 - ln4}{2}$

D. $\frac{3 - ln4}{2}$

4. Simplify $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$

A. 5

B. 0

C. 2

D. -4

5. If $z = \frac{2+it}{t+2i}$ and given that t is a real variable. The locus of z as t varies is given by:

A. A circle centred at (2, 0) and radius 1 unit

B. A circle centred at (0, 0) and radius 1 unit.

C. A circle centred at (0, 0) and radius 2 units.

D. A circle centred at (2, 0) and radius 2 units.

6. If $y = \frac{1}{2}(e^x - e^{-x})$ find x in terms of y.

A. $x = \ln(y - \sqrt{y^2 + 1})$

B. $x = \ln(-y + \sqrt{y^2 + 1})$

C. $x = \ln(-y - \sqrt{y^2 + 1})$

D. $x = \ln(y + \sqrt{y^2 + 1})$

7. A point P lies on the ellipse $\frac{X^2}{9} + \frac{Y^2}{4} = 1.$

The perpendicular from P meets the directrix at M.

The focus of the ellipse is S. Find the value of the ratio $\frac{PS}{PM}$

A. $\frac{\sqrt{5}}{3}$

B. $\frac{\sqrt{3}}{5}$

C. $\frac{\sqrt{5}}{9}$

D. $\frac{2}{3}$

8. Find the volume generated when a circle of radius a units is rotated about its vertical tangent.

A. $2\pi^2 a^3$

B. $\frac{8\pi^2 a^3}{3}$

C. $\frac{4\pi a^3}{3}$

D. $4\pi^2 a^3$

9. From the digits 0, 1, 2, 3, ………, 9 two digits are selected without replacement.

If they are both odd digits, what is the probability that their sum is greater than 10?

A. $\frac{3}{5}$

B. $\frac{1}{4}$

C. $\frac{2}{5}$

D. $\frac{3}{10}$

10. Find the exact value of $\int_0^1 xe^{-2x}dx$

A. $\frac{-1}{2}e^{-2}(e^2 + 1)$

B. $\frac{1}{2}e^{-2}(e^2 - 3)$

C. $\frac{1}{4}e^{-2}(e^2 - 1)$

D. $\frac{1}{4}e^{-2}(e^2 - 3)$

Section II (90 marks)

Attempt Questions 11–16.

Allow about 2 hours and 45 minutes for this section.

Answer each question in the appropriate writing booklet.

Extra writing booklets are available.

Your responses should include relevant mathematical reasoning, working and formulae.

Question 11 (15 marks) marks

(a) Express $\frac{9-7i}{1+i} - \frac{5}{2+i}$ in the form $m + ni$. **2**

(b) If $f(x) = \sqrt{4 - x^2}$ then graph neatly on separate number planes:

(Each graph should be approx $\frac{1}{3}$ of a page showing all important features)

(i) $y = f(x)$ **2**

(ii) $y = \frac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = xf(x)$ **2**

(c) The roots of the cubic equation $2x^3 + 4x^2 - 6x + 1 = 0$ are α, β and γ.

Find the equation whose roots are $2\alpha + 1, 2\beta + 1$ and $2\gamma + 1$. **2**

(d) Find the coordinates of the point/s on the curve $x^2 + y^2 = xy + 3$, where the tangent/s is horizontal. **3**

<u>Question 12</u> (15 marks) marks

(a) Shade neatly the region on the Argand Diagram represented by

$$-2 \leq Im(z) \leq 1 \quad \cup \quad \frac{-\pi}{3} \leq Arg z \leq \frac{\pi}{4}$$ **2**

(b) Given the line $y = mx + 5$ and the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Find the value/s of *m* such that the given line is a tangent to the ellipse. **3**

(c) The region bounded by $y = \sqrt{9 - x^2}$, the x-axis and the line $x = 2$ is rotated about the y-axis.

(i) Using the slices method, show that the volume of a slice is given by

$$\delta V = \pi(5 - y^2)\delta y$$ **2**

(ii) Hence find the exact volume of the solid of revolution. **2**

(d) Evaluate the integral $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{cos^3 x}{sin^2 x}\, dx$ leaving your answer in exact form. **3**

(e) Differentiate the expression $\tan^{-1} 5x + \tan^{-1}\frac{1}{5x}$ and hence find the value of the expression for $x < 0$. **3**

Question 13 (15 marks) marks

(a) The base of a solid is a circle of radius 6 units.

Cross sections of the solid by planes perpendicular to its base are equilateral triangles.

(i) Show that the solid's volume is given by $V = \int_{-6}^{6} \sqrt{3}(36 - x^2)dx$. **3**

(ii) Hence find the exact volume of this solid. **1**

(b) In a class of 15 girls, one girl is chosen to be the referee and the other girls play 7 a side soccer. In how many ways can the referee and teams be chosen? **2**

(c) A sequence of numbers is such that $T_1 = 7$, $T_2 = 29$ and $T_{n+2} = 7T_{n+1} - 10T_n$.

Prove by Mathematical Induction that $T_n = 2^n + 5^n$ for $n \geq 1$. **4**

(d) Given that w is a complex cube root of unity,

evaluate exactly $1 + w + w^2 + w^3 + \ldots\ldots. + w^{1001}$ **2**

(e) (i) Express $w = -2 + 2\sqrt{3}\,i$ in mod-arg form. **1**

(ii) Given that w is a root of $w^3 - a^3 = 0$, find the exact value/s of a. **2**

Question 14 (15 marks) marks

(a) (i) Find the equation of the tangent at $(2t, \frac{1}{t})$ to xy = 2 in general form. **2**

(ii) Find the product of the perpendiculars from (2, 2) and (-2, -2) to the tangent. **2**

(b) (i) Show that $I_n = e - nI_{n-1}$ if $I_n = \int_1^e (lnx)^n \, dx$ for $n \geq 0$. **3**

(ii) Hence evaluate I_5 **2**

(c) (i) Prove that $\cos 4\theta = 8\cos^4\theta - 8cos^2\theta + 1$ **2**

(ii) Hence solve the equation $16x^4 - 16x^2 + 1 = 0$. **2**

(iii) Hence show that the exact value of $(\cos\frac{\pi}{12} + \cos\frac{5\pi}{12})$ is $\sqrt{\frac{3}{2}}$. **2**

Question 15 (15 marks) marks

(a) Use the method of cylindrical shells to find the volume of the solid obtained when the region bounded by $y = 4x^2 - x^4$, $y \geq 0, x \geq 0$ and $x \leq 2$ is rotated about the *y*-axis. **4**

(b) A rock of mass 5kg is falling through water.

The resistance of the water gives an upward force of 0.25N and the buoyancy of the water provides a further upwards force of 6N. Taking *g* as 10m/s^2, find the acceleration of the rock. **1**

(c) A particle is thrown vertically upwards with a velocity of *U*m/s. It experiences a resistance which is proportional to mv.

(i) Show that its acceleration is given by $\ddot{x} = -g - kv$, where *k* is a positive constant and g is the acceleration due to gravity. **1**

(ii) Find when the particle reaches its maximum height. **3**

(iii) Find the greatest height, *H*. **3**

(d) Two circles touch externally at *A*.

A common tangent touches the circles at *M* and *N* respectively.

Find the size of ∠*MAN*, giving reasons. **3**

Question 16 (15 marks) marks

(a) (i) Calculate the area of the ellipse $x^2 + 16y^2 = 16$. **1**

(ii) A solid has the ellipse $x^2 + 16y^2 = 16$ as its base. Cross-sections of the solid perpendicular to its base and parallel to the *y*-axis are rectangles of height 6 units.

(α) Draw this information, clearly showing a typical slice. **2**

(β) Show that the expression for the volume of this slice is given by:

$$\delta V = 3\sqrt{16 - x^2}\delta x.$$ **2**

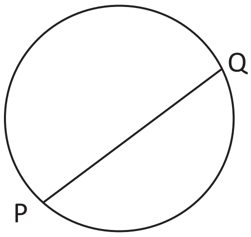
(γ) Calculate the exact volume of the solid. **1**

(b) *O* is the centre of the circle. *P* is a fixed point on the circumference of the circle.

Q is a variable point on the circle's circumference.

With *OP* as a diameter a semi-circle is drawn.

PQ meets this semi-circle at *R*. Prove that *R* is always the midpoint of *PQ*. **2**



(d) A particle of unit mass moves in a straight line against a resistance numerically equal to $v + v^3$, where v is its velocity. Initially the particle is at the origin and is travelling with velocity K, where $K > 0$.

(i) Show that v is related to the displacement x by $x = \tan^{-1}\left(\frac{K-v}{1+Kv}\right)$. **3**

(ii) Show that the time t which has elapsed when the particle is travelling with velocity v, is given by $t = \frac{1}{2} ln\left[\frac{K^2(1+v^2)}{v^2(1+K^2)}\right]$. **4**

End of Examination

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Ext 2 2019 HSC Trial

Section 1 - Multiple Choice

1. $(5+2i)^2 - (3-2i)^2$

$= 25 + 20i + 4i^2 - (9 - 12i + 4i^2)$

$= 25 + 20i - 4 - (9 - 12i - 4)$

$= 21 + 20i - (5 - 12i)$

$= 21 + 20i - 5 + 12i$

$= 16 + 32i$

$= 16(1 + 2i)$ (C)

2. $P(x) = x^4 + 2x^3 - 12x^2 + 14x - 5$ has triple root

$P'(x) = 4x^3 + 6x^2 - 24x + 14$

$P''(x) = 12x^2 + 12x - 24$

$= 12(x^2 + x - 2)$

Solving $P''(x) = 0$: $12(x^2 + x - 2) = 0$

$(x+2)(x-1) = 0$

$\therefore x = -2, 1$

$P(-2) = (-2)^4 + 2(-2)^3 - 12(-2)^2 + 14(-2) - 5 = -81$

$P(1) = (1)^4 + 2(1)^3 - 12(1) + 14(1) - 5 = 0$

$\therefore x = 1$ is triple root

$\alpha + \beta + \gamma + \delta = 1 + 1 + 1 + \delta = -2$

$\therefore \delta = -5$

$\therefore$ roots are $1, 1, 1, -5$ (D)

3. $\displaystyle\int_0^1 \frac{x^2}{x+1}\,dx = \int_0^1 \frac{x^2-1}{x+1} + \frac{1}{x+1}\,dx$

$\displaystyle = \int_0^1 x - 1 + \frac{1}{x+1}\,dx$

$\displaystyle = \left[\frac{x^2}{2} - x + \ln(x+1)\right]_0^1$

$\displaystyle = \left[\frac{1}{2} - 1 + \ln(2)\right] - \left[0 - 0 + \ln(1)\right]$

$\displaystyle = \frac{2\ln(2) - 1}{2}$

$\displaystyle = \frac{\ln(4) - 1}{2}$ (B)

1.16

8:55

9:20

4. $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}$

$= \frac{\sin 3x \cos x - \sin x \cos 3x}{\sin x \cos x}$

$= \frac{\sin(3x - x)}{\sin x \cos x}$ $(\because \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$= \frac{\sin 2x}{\sin x \cos x}$

$= \frac{2\sin 2x}{\sin 2x}$ $(\because \sin 2x = 2\sin x\cos x)$

$= 2$ (c)

5. $z = \frac{2+it}{t+2i} \times \frac{t-2i}{t-2i}$

$= \frac{2t - 4i + it^2 - 2i^2 t}{t^2 - 4i^2}$

$= \frac{(4t) + i(t^2-4)}{t^2+4}$

$X = \frac{4t}{t^2+4}$ $Y = \frac{t^2-4}{t^2+4}$

$X^2 + Y^2 = \left(\frac{4t}{t^2+4}\right)^2 + \left(\frac{t^2-4}{t^2+4}\right)^2$

$= \frac{16t^2 + t^4 - 8t^2 + 16}{(t^2+4)^2}$

$= \frac{t^4 + 8t^2 + 16}{(t^2+4)^2}$

$= \frac{(t^2+4)^2}{(t^2+4)^2}$ (B)

$\therefore X^2 + Y^2 = 1$ $\Rightarrow$ circle centre (0,0) radius 1 unit

6. $y = \frac{1}{2}(e^{x} - e^{-x})$

$2y = e^{x} - e^{-x}$ Multiply Both sides by e^{x}

$e^{2x} - 2ye^{x} - 1 = 0$

This is a quadratic equation in e^{x}:

$$e^{x} = \frac{2y \pm \sqrt{4y^2 - 4(1)(-1)}}{2(1)}$$

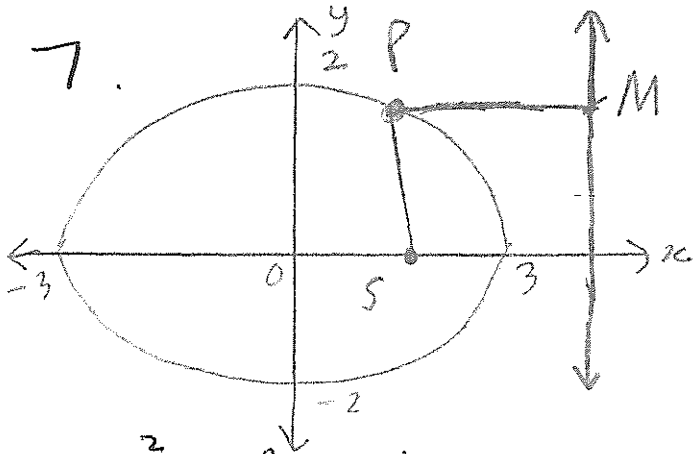
$$= \frac{2y \pm \sqrt{4y^2+4}}{2}$$

$$= \frac{2y \pm 2\sqrt{y^2+1}}{2}$$

$$e^{x} = y \pm \sqrt{y^2+1}$$ Since $e^{x} > 0$

Taking logs of both sides: $x = \ln(y + \sqrt{y^2+1})$ D

7.



$\frac{x^2}{9} + \frac{y^2}{4} = 1$

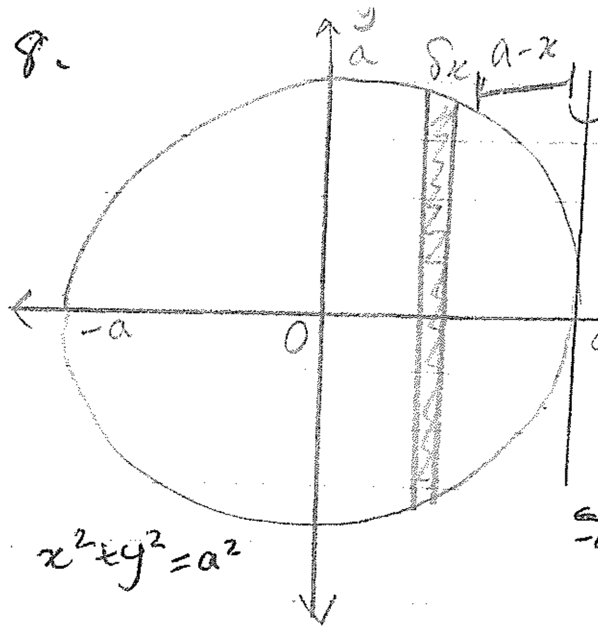
$$\frac{PS}{PM} = e$$

$$= \sqrt{1 - \frac{b^2}{a^2}}$$

$$= \sqrt{1 - \frac{4}{9}}$$

$$= \frac{\sqrt{5}}{3}$$ A

8.



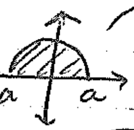
$x^2 + y^2 = a^2$

$$\delta V = 2\pi(a-x)\cdot(2y)\,\delta x$$

$$= 4\pi(a-x)y\,\delta x$$

$$= 4\pi(a-x)(\sqrt{a^2-x^2})\,\delta x$$

$$\therefore V = 4\pi\int_{-a}^{a} a\sqrt{a^2-x^2} - x\sqrt{a^2-x^2}\,dx$$



$$= 4\pi a \times \left(\frac{1}{2} \times \pi \times a^2\right)$$

$$= 2\pi^2 a^3 \text{ units}^3$$

A

Odd function

9. $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$

Both odd $\Rightarrow$ $1, 3, 5, 7, 9$

sum greater than 10:

① $9+7$

② $9+5$

③ $9+3$

④ $7+5$

$\therefore$ probability $= \dfrac{4}{{}^5C_2}$

$= \dfrac{2}{5}$ C

10. $\int_0^1 x e^{-2x}\,dx$

$u = x \qquad v = -\frac{1}{2}e^{-2x}$

$u' = 1 \qquad v' = e^{-2x}$

$$= \left[-\frac{1}{2}x e^{-2x}\right]_0^1 - \int_0^1 \left(-\frac{1}{2}\right)e^{-2x}\,dx$$

$$= -\frac{1}{2}\left((1)e^{-2} - 0\right) + \frac{1}{2}\int_0^1 e^{-2x}\,dx$$

$$= -\frac{1}{2}e^{-2} - \frac{1}{4}\left[e^{-2x}\right]_0^1$$

$$= -\frac{1}{2}e^{-2} - \frac{1}{4}\left(e^{-2} - 1\right)$$

$$= -\frac{1}{2}e^{-2} - \frac{1}{4}e^{-2} + \frac{1}{4}$$

$$= -\frac{3}{4}e^{-2} + \frac{1}{4}$$

$$= \frac{1}{4}e^{-2}\left(e^{2} - 3\right)$$

D

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Level: Extension 2

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QUESTION: 11 | Markers Comments

(a) $\dfrac{9-7i}{1+i} - \dfrac{5}{2+i}$

$= \left(\dfrac{9-7i}{1+i} \times \dfrac{1-i}{1-i}\right) - \left(\dfrac{5}{2+i} \times \dfrac{2-i}{2-i}\right)$

$= \dfrac{9-9i-7i+7i^2}{1-i^2} - \dfrac{(10-5i)}{4-i^2}$

1 for each fraction

$= \dfrac{9-16i-7}{1+1} - \dfrac{(10-5i)}{4+1}$

$= \dfrac{2-16i}{2} - \dfrac{(10-5i)}{5}$

$= 1-8i-2+i$

$= -1-7i$

or

$= \dfrac{18+9i-14i+7-5-5i}{(1+i)(2+i)}$

$= \dfrac{20-10i}{1+3i} \times \dfrac{1-3i}{1-3i}$ 1

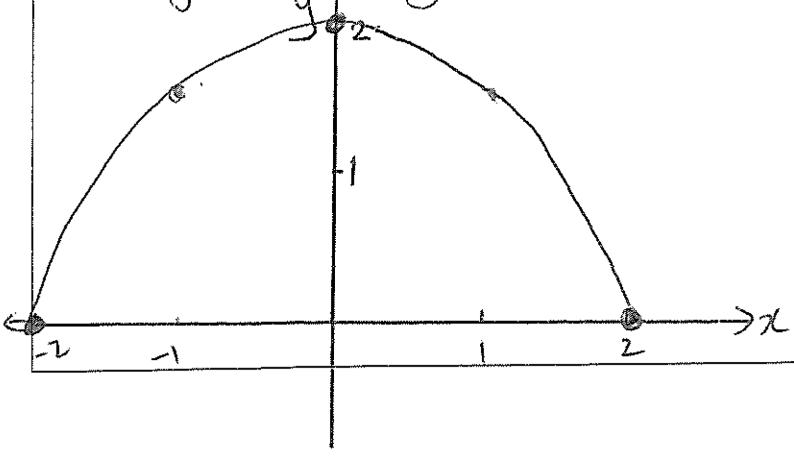
* A lot of careless algebraic errors.

$= \dfrac{20-30-10i-60i}{10}$

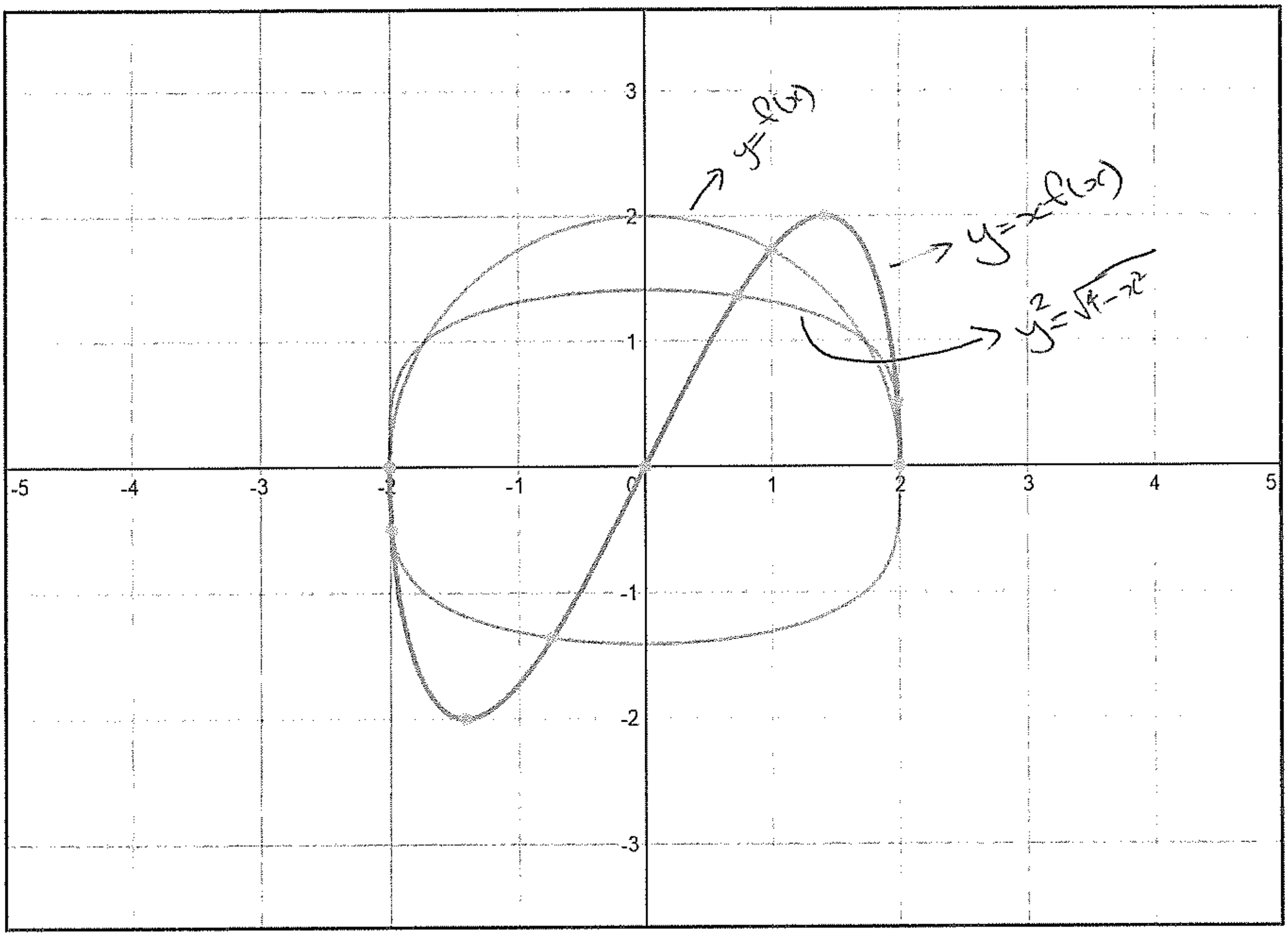
$= \dfrac{-10-70i}{10}$

$= -1-7i$ 1

(b) (i) $y=f(x)$ $y=\sqrt{4-x^2}$



* should use the same scale on x and y-axis, so it looks like a semi-circle not a concave down parabola.


$y=f(x)$
$y=xf(x)$
$y^2=\sqrt{4-x^2}$
3
2
1
-1
-2
-3
-5
-4
-3
-2
-1
0
1
2
3
4
5

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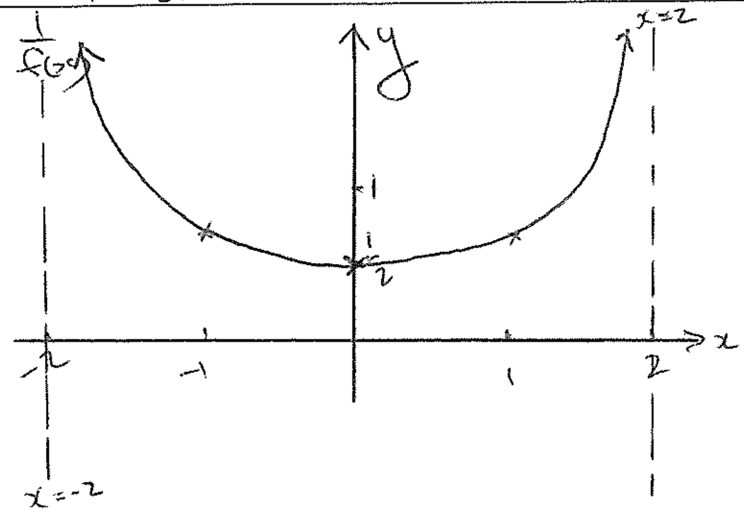
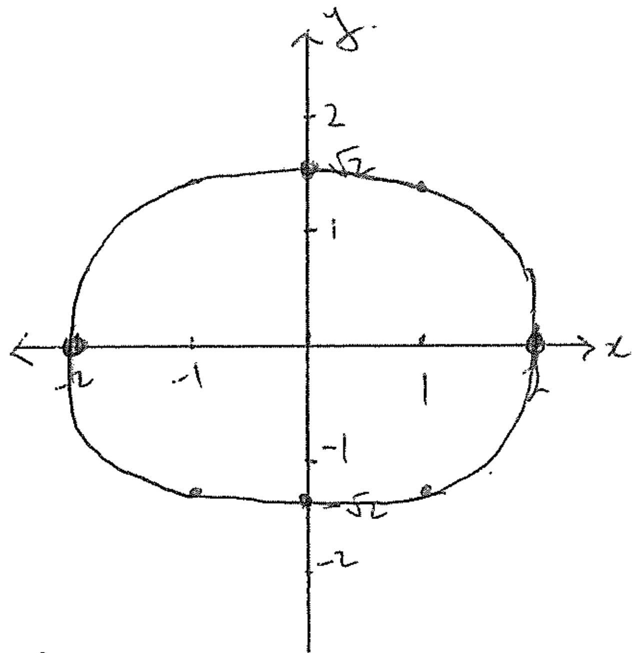
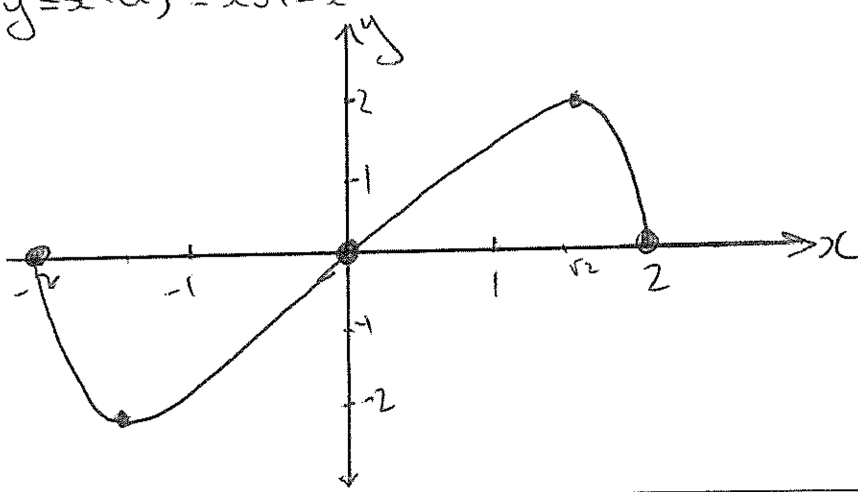
QUESTION: 11 cont.	Markers Comments
(ii) $y = \frac{1}{f(x)}$ 	(1) for the 2 vertical asymptotes (1) for shape and $(0, \frac{1}{2})$ being clearly labelled.
(iii) $y^2 = f(x)$ 	(1) for y intercepts $\pm\sqrt{2}$, x-intercepts of ± 2. (1) for shape *not a circle
(iv) $y = xf(x) = x\sqrt{4-x^2}$ 	(1) for shape (1) for max. at $(\sqrt{2}, 2)$ or vicinity (graph is not symmetrical about the y-axis).

Examination:

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QUESTION: 11 cont.	Markers Comments

(c) $P(x) = 2x^3 + 4x^2 - 6x + 1 = 0$

has roots of α, β, γ

let $y = 2x+1$

$\therefore x = \frac{y-1}{2}$ sub. into $P(x)$

$2\left(\frac{y-1}{2}\right)^3 + 4\left(\frac{y-1}{2}\right)^2 - 6\left(\frac{y-1}{2}\right) + 1 = 0$

$\frac{2(y^3 - 3y^2 + 3y - 1)}{8} + \frac{4(y^2 - 2y + 1)}{4} - 3(y-1) + 1 = 0$

$y^3 - 3y^2 + 3y - 1 + 4y^2 - 8y + 4 - 12y + 12 + 4 = 0$ (1)

$y^3 + y^2 - 17y + 19 = 0$ (1)

$\therefore x^3 + x^2 - 17x + 19 = 0$ (dummy variable)

* generally well done, only minor algebraic errors.

(d) $x^2 - xy + y^2 - 3 = 0$

Differentiate implicitly with respect to x

$2x - \left((1)(y) + (x)\frac{dy}{dx}\right) + 2y\left(\frac{dy}{dx}\right) - 0 = 0$

$2x - y + (-x + 2y)\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx}(-x + 2y) = -2x + y$

$\frac{dy}{dx} = \frac{-(2x-y)}{-x+2y}$ (1)

* badly done

horizontal tangent when $\frac{dy}{dx} = 0$

ie $-2x + y = 0$

$\therefore y = 2x$ (1)

sub $y = 2x$ into eqn.

$x^2 + (2x)^2 = x(2x) + 3$

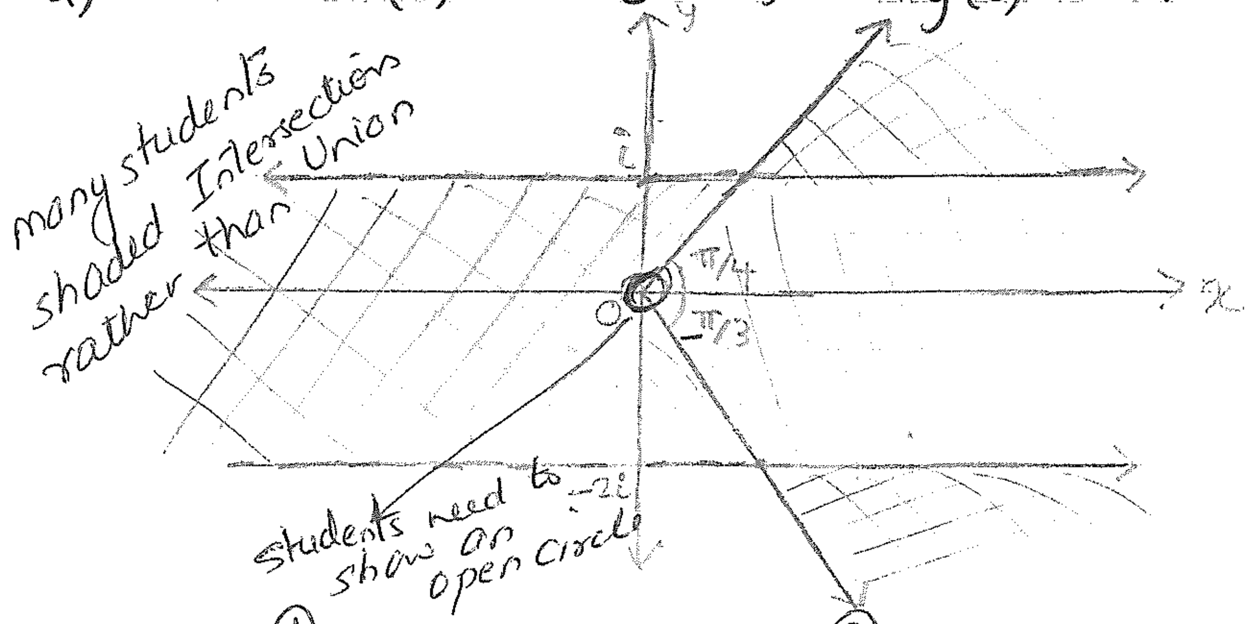
$x^2 + 4x^2 = 2x^2 + 3$

$3x^2 = 3 \quad x^2 = 1 \quad x = \pm 1$

$\therefore$ coordinates are $(1,2)$ and $(-1,-2)$ (1)

Question 12

a) $-2 \le \mathrm{Im}(z) \le 1 \;\cup\; -\frac{\pi}{3} \le \arg(z) \le \pi/4$



b) ① $y = mx + 5$ is tangent to ② $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Sub ① in ② $\longrightarrow$ $\frac{x^2}{16} + \frac{(mx+5)^2}{9} = 1$

$9x^2 + 16(mx+5)^2 = 144$

$9x^2 + 16(m^2x^2 + 10mx + 25) = 144$

$x^2(16m^2+9) + 160mx + 400 - 144 = 0$

$x^2(16m^2+9) + 160mx + 256 = 0$ (1)

$\Delta = (160m)^2 - 4(16m^2+9)(256)$

$\boxed{\text{Since } y = mx+5 \text{ is a tangent, } \Delta = 0}$: this condition is mandatory

$(160m)^2 - 4(16m^2+9)(256) = 0$ (1)

$25600m^2 - (64m^2+36)(256) = 0$

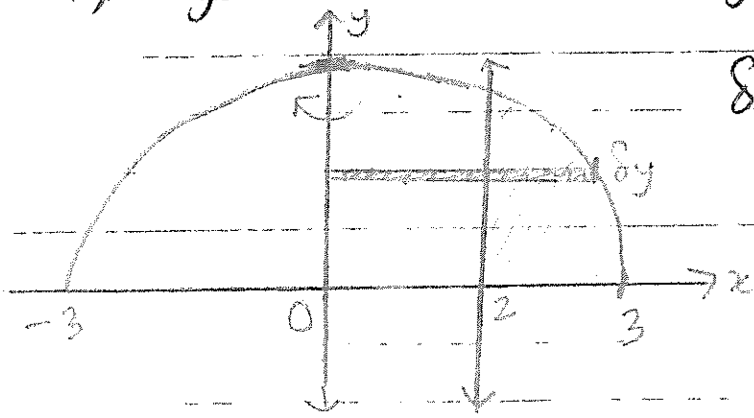
$(\div 256)$ $100m^2 - 64m^2 - 36 = 0$

$25m^2 - 16m^2 - 9 = 0$

$9m^2 = 9$

$m = \pm 1$ (1)

c) $y = \sqrt{9-x^2} \rightarrow 9-x^2 = y^2 \quad \therefore x^2 = 9-y^2$ ①



$$\delta V = \pi(r_2^2 - r_1^2)\delta y$$
$$= \pi(x^2 - 2^2)\delta y$$
$$= \pi(x^2-4)\delta y$$
$$= \pi((9-y^2)-4)\delta y \quad \text{using ①}$$
$$= \pi(5-y^2)\delta y$$

$$V = \lim_{\delta y \to 0} \sum_{y=0}^{y=\sqrt{5}} \pi(5-y^2)\delta y$$

Note: At $x=2$
$y = \sqrt{9-x^2} = \sqrt{5}$

$$= \pi\int_0^{\sqrt{5}} (5-y^2)\,dy$$
$$= \pi\left[5y - \frac{y^3}{3}\right]_0^{\sqrt{5}}$$

majority wrote limits for x while integrating with respect to y.

$$= \pi\left[5\sqrt{5} - \frac{5\sqrt{5}}{3}\right] - 0$$
$$= \frac{10\pi\sqrt{5}}{3} \text{ units}^3$$

d) $$\int_{\pi/6}^{\pi/4} \frac{\cos^3 x}{\sin^2 x}\,dx = \int_{\pi/6}^{\pi/4} \frac{\cos^2 x \cdot \cos x\,dx}{\sin^2 x}$$
$$= \int_{\pi/6}^{\pi/4} \frac{(1-\sin^2 x)\cdot\cos x\,dx}{\sin^2 x}$$

Let $u = \sin x$
$du = \cos x\,dx$
At $x = \frac{\pi}{4}$, $u = 1/\sqrt{2}$
At $x = \pi/6$, $u = 1/2$

$$= \int_{1/2}^{1/\sqrt{2}} \frac{1-u^2}{u^2}\,du$$
$$= \int_{1/2}^{1/\sqrt{2}} \frac{1}{u^2} - 1\,du$$
$$= \left[-\frac{1}{u} - u\right]_{1/2}^{1/\sqrt{2}}$$
$$= \left[-\sqrt{2} - \frac{1}{\sqrt{2}}\right] - \left[-2 - \frac{1}{2}\right] = \frac{5}{2} - \frac{3\sqrt{2}}{2} = \frac{5-3\sqrt{2}}{2}$$

Many students did algebraic error while carrying out the limits

12 e) $\frac{d}{dx}\left[\tan^{-1} 5x + \tan^{-1}\left(\frac{1}{5x}\right)\right]$

$$= \left[\frac{1}{1+(5x)^2} \times 5\right] + \left[\frac{1}{1+\left(\frac{1}{5x}\right)^2} \times \frac{d}{dx}\left(\frac{1}{5x}\right)\right]$$

$$= \frac{5}{1+25x^2} + \left(\frac{1}{1+\frac{1}{25x^2}} \times \frac{-1}{5x^2}\right)$$

$$= \frac{5}{1+25x^2} + \left(\frac{-1}{5x^2+\frac{1}{5}}\right)$$

$$= \frac{5}{1+25x^2} + \left(\frac{-5}{25x^2+1}\right)$$

$$= 0$$

only few could show that $\frac{d}{dx}\left(\tan^{-1} 5x + \tan^{-1}\frac{1}{5x}\right) = 0$

forget chain rule.

$$\therefore \tan^{-1}(5x) + \tan^{-1}\left(\frac{1}{5x}\right) = \int 0 \, dx = C$$

sub $x = -1$:

$$LHS = \tan^{-1}(-5) + \tan^{-1}\left(\frac{-1}{5}\right) = -\pi/2$$

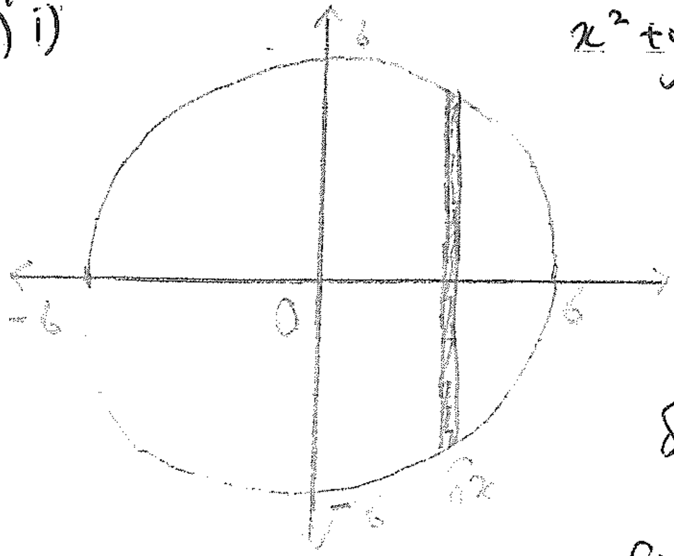
ie $C = -\pi/2$

$$\therefore \tan^{-1}(5x) + \tan^{-1}\left(\frac{1}{5x}\right) = -\frac{\pi}{2} \quad \boxed{\text{when } x < 0}$$

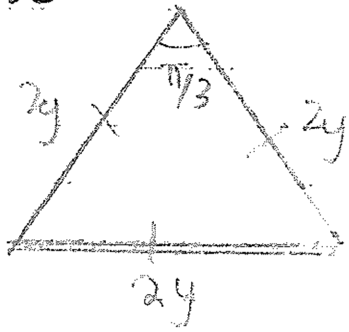
read carefully.

Question 13

a) i)



$x^2+y^2=36$



$$\delta V = \frac{1}{2}\times(2y)\times(2y)\times\sin\frac{\pi}{3}\cdot\delta x$$
$$= \sqrt{3}\,y^2\cdot\delta x$$
$$\delta V = \sqrt{3}(36-x^2)\cdot\delta x$$

A few students didn't use the formula, but did Guess and check method.

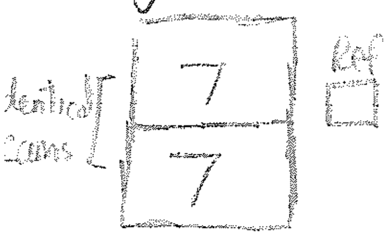
$$V = \lim_{\delta x\to 0}\sum_{x=-6}^{x=6}\sqrt{3}(36-x^2)\,\delta x$$

$$\therefore V = \sqrt{3}\int_{-6}^{6} 36-x^2\,dx$$

ii)
$$= \sqrt{3}\left[36x-\frac{x^3}{3}\right]_{-6}^{6}$$
$$= \sqrt{3}\left[36(6)-\frac{6^3}{3}\right]-\sqrt{3}\left[36(-6)-\frac{(-6)^3}{3}\right]$$
$$= \underline{288\sqrt{3}\text{ units}^3}$$

well done

b) 15 girls



Ref — 2 rooms which are identical

$$\text{total ways} = \frac{{}^{15}C_1\times{}^{14}C_7\times{}^{7}C_7}{2!}$$

Some students forgot to divide by 2!

$= 25740$ ways

c) Prove by mathematical induction $T_n = 2^n+5^n$ for $n\geq 1$

Given $T_1=7$, $T_2=29$ $\quad T_{n+2}=7T_{n+1}-10T_n$

Step 1: Prove true for $n=1$ and $n=2$:

$LHS = T_1 = 7$ | $LHS = T_2 = 29$

$RHS = 2^1+5^1=7$ | $RHS = 2^2+5^2=29$

$\therefore$ true for $n=1$ and $n=2$

Step 2: Assume true for $n=k$, where k is an integer $(k \leq n)$:

$T_k = 2^k + 5^k$

$T_{k-1} = 2^{k-1} + 5^{k-1}$

Step 3: Prove true for $n=k+1$:

ie $T_{k+1} = 2^{k+1} + 5^{k+1}$

$LHS = T_{k+1}$

$= 7T_k - 10T_{k-1}$ (using $T_{n+2} = 7T_{n+1} - 10T_n$)

$= 7(2^k+5^k) - 10(2^{k-1}+5^{k-1})$ using step 2.

$= 7\times2^k + 7\times5^k - 10\times2^{k-1} - 10\times5^{k-1}$

$= 7\times2^k + 7\times5^k - (5\times2.2^{k-1}) + (2\times5.5^{k-1})$

$= 7\times2^k + 7\times5^k - 5\times2^k - 2\times5^k$

$= 2\times2^k + 5\times5^k$

$= 2^{k+1} + 5^{k+1}$

$= RHS$ Hence Proved.

well done!!

Step 4: By the Principle of Mathematical Induction, the result is true for all integers $n \geq 1$.

d) $\omega^3 = 1$ ie $\omega = cis\left(\frac{2k\pi}{3}\right)$, where $k = 0, \pm1$

$\omega = 1, cis\frac{2\pi}{3}, cis\left(-\frac{2\pi}{3}\right)$

ω is complex so $\omega \neq 1$

$\omega^3 - 1 = 0$

$(\omega-1)(\omega^2+\omega+1) = 0$

some st

Since $\omega \neq 1$ $\omega^2+\omega+1 = 0$

$1+\omega+\omega^2+\omega^3+\dots\dots+\omega^{1001}$

$= (1+\omega+\omega^2) + (\omega^3+\omega^4+\omega^5) + \dots + \omega^{1000} + \omega^{1001}$

$= (1+\omega+\omega^2) + (1+\omega+\omega^2) + \dots + (1+\omega+\omega^2)$

$= 0 + 0 + \dots + (1+\omega+\omega^2) = 0$

$1002 \div 3$

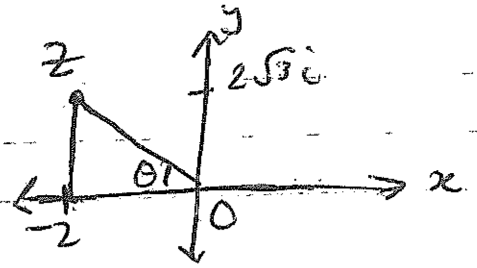
$= 334$

since $\omega^3 = 1$

$\omega^4 = \omega^3\times\omega = \omega$

A better method :

$w = -2 + 2\sqrt{3}i$



$$|w| = \sqrt{(-2)^2 + (2\sqrt{3})^2}$$
$$= \sqrt{4 + 12}$$
$$= 4 \text{ units}$$

$$\arg(w) = \pi - \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right)$$
$$= \pi - \pi/3$$
$$= 2\pi/3$$

$$\therefore w = 4\,\text{cis}\left(\frac{2\pi}{3}\right)$$

A few students forgot to find the rest of two root

Solving $a^3 = w^3 = \left(4\,\text{cis}\,\frac{2\pi}{3}\right)^3$

$$= 64\,\text{cis}(2\pi)$$

$\underline{a^3 = 64}$

$\therefore a = 4\,\text{cis}\left(\frac{2k\pi}{3}\right)$, where k is an integer

$k = 0, \pm 1$

$$\therefore a = 4\,\text{cis}(0),\quad 4\,\text{cis}\left(\frac{2\pi}{3}\right),\quad 4\,\text{cis}\left(\frac{-2\pi}{3}\right)$$
$$= 4,\quad (-2+2\sqrt{3}i),\quad (-2-2\sqrt{3}i)$$

w $\qquad \bar{w}$

Another Method :

d) $1 + w + w^2 + \cdots + w^{1001}$

$(1+w+w^2) + w^3(1+w+w^2) + \cdots + w^{996}(1+w+w^2)$

$+ w^{999}(1+w+w^2) = 0$

Some students missed last term

Another Method :

$$1 + w + w^2 + \cdots + w^{1001} = \frac{w^{1002} - 1}{w - 1}$$

$\Rightarrow (w-1)(1+w+\cdots+w^{1001}) = w^{1002} - 1$

Sum of roots $= 0$ $\quad \therefore (w-1)(1+w+\cdots+w^{1001}) = 0$

Since $w \neq 1$, $\quad 1 + w + \cdots + w^{1001} = 0$.

e) $w = -2 + 2\sqrt{3}i$

$w^3 - a^3 = 0$

$(w-a)(w^2 + aw + a^2) = 0$

ie $w = a$ or $w^2 + aw + a^2 = 0$

$\therefore \underline{\underline{a = -2+2\sqrt{3}i}}$

Quadratic in a:

$a = \frac{-w \pm \sqrt{w^2 - 4(w^2)(1)}}{2(1)}$

$= \frac{-w \pm \sqrt{-3w^2}}{2}$

$= \frac{-w \pm \sqrt{3}iw}{2}$

$\therefore a = \frac{-w + \sqrt{3}iw}{2}$ or $\frac{-w - \sqrt{3}iw}{2}$

ie $a = \frac{-(-2+2\sqrt{3}i) + \sqrt{3}i(-2+2\sqrt{3}i)}{2}$

$= \frac{2 - 2\sqrt{3}i - 2\sqrt{3}i + 6i^2}{2}$

$= \frac{-4 - 4\sqrt{3}i}{2}$

$\therefore \underline{\underline{a = -2 - 2\sqrt{3}i}}$

$a = \frac{-(-2+2\sqrt{3}i) - \sqrt{3}i(-2+2\sqrt{3}i)}{2}$

$= \frac{2 - 2\sqrt{3}i + 2\sqrt{3}i - 6i^2}{2}$

$= \frac{8}{2}$

$= 4$

$\therefore a = 4, -2+2\sqrt{3}i, -2-2\sqrt{3}i$

Examination: Trial HSC

Level: Ext 2

Year: 2019

QUESTION: 14

	Markers Comments
(a)(i) $xy = 2$ $\therefore y = \frac{2}{x}$ $\frac{dy}{dx} = \frac{-2}{x^2}$ at $x = 2t$, $\frac{dy}{dx} = \frac{-2}{(2t)^2} = \frac{-1}{2t^2}$	really worrying that most students chose to use implicit differentiation, ✱ 1
eqn. of tangent is: $y - \frac{1}{t} = \frac{-1}{2t^2}(x - 2t)$ $2t^2y - 2t = -(x - 2t)$ $2t^2y - 2t = -x + 2t$ $\therefore x + 2t^2y - 4t = 0$ ... (a)	✱ A lot of students didn't know what general form was.... 1
(ii) perpendicular distance from $(2,2)$ to ... (a) $d_1 = \frac{\lvert 2 + 2(2t^2) - 4t \rvert}{\sqrt{1+4t^4}}$ $= \frac{\lvert 4t^2 - 4t + 2 \rvert}{\sqrt{1+4t^4}}$ from $(-2,-2)$ to ... (a) $d_2 = \frac{\lvert -2 + -2(2t^2) - 4t \rvert}{\sqrt{1+4t^4}}$ $d_2 = \frac{\lvert -2 - 4t^2 - 4t \rvert}{\sqrt{1+4t^4}}$	1 for correct perpendicular distance simplified.
$\therefore$ prod. of d_1 and $d_2 = \frac{4(2t^2 - 2t + 1)(2t^2 + 2t + 1)}{1 + 4t^4}$ $= \frac{4(4t^4 + 4t^3 + 2t^2 - 4t^3 - 4t^2 - 2t + 2t^2 + 2t + 1)}{1+4t^4}$ $= \frac{4(1 + 4t^4)}{1 + 4t^4}$ $= 4$	1

Examination:

Level: Ext 2

Year: 2019

pg 2

QUESTION: 14	Markers Comments
(b)(i) $I_n = \int_1^e (\ln x)^n . 1 . dx$ $\quad u = (\ln x)^n \quad v = x$ $\frac{du}{dx} = \frac{n(\ln x)^{n-1}}{x} \quad \frac{dv}{dx} = 1$	} (1)
$\therefore I_n = \left[x(\ln x)^n\right]_1^e - \int_1^e \frac{n(\ln x)^{n-1}}{x} \times x \, dx$	(1)
$= e(\ln e)^n - 1(\ln 1)^n - n\int_1^e (\ln x)^{n-1} dx$	(1)
$= e - 0 - nI_{n-1}$ $= e - nI_{n-1}$	* need to show the line of substitution as it's a show question; worth 3 marks!!
(ii) $I_5 = e - 5I_4$ $= e - 5(e - 4I_3)$ $= e - 5e + 20(e - 3I_2)$ $= -4e + 20e - 60(e - 2I_1)$ $= 16e - 60e + 120(e - I_0)$	
now $I_0 = \int_1^e 1 \, dx$ $= [x]_1^e$ $= e - 1$	(1)
$\therefore I_5 = -44e + 120e - 120I_0$ $= -44e + 120e - 120(e-1)$ $= 76e - 120e + 120$ $= -44e + 120$	(1)

Examination:

Level: Ext 2.

Year: 2019

pg 3

QUESTION: 14 cont.	Markers Comments

(c) (i) method 1

$(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ (by DeMoivre's Theorem)

or $(\cos\theta + i\sin\theta)^4 = \cos^4\theta + 4i\sin\theta\cos^3\theta + 6i^2\sin^2\theta\cos^2\theta + 4i^3\sin^3\theta\cos\theta + i^4\sin^4\theta$ (1)

$= \cos^4\theta + 4i\sin\theta\cos^3\theta - 6\sin^2\theta\cos^2\theta - 4i\sin^3\theta\cos\theta + \sin^4\theta$

equating real parts:

$\therefore \cos 4\theta = \cos^4\theta - 6\sin^2\theta\cos^2\theta + \sin^4\theta$

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= \cos^4\theta - 6\cos^2\theta + 6\cos^4\theta + 1 - 2\cos^2\theta + \cos^4\theta$ (1)

$\therefore \cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$

or

method 2 LHS $= \cos 4\theta$

$\cos 4\theta = \cos(2\theta + 2\theta)$

$= \cos^2 2\theta - \sin^2\theta$

$= 2\cos^2 2\theta - 1$ (1)

$= 2(2\cos^2\theta - 1) - 1$

$= 2(4\cos^4\theta - 4\cos^2\theta + 1) - 1$ (1)

$= 8\cos^4\theta - 8\cos^2\theta + 2 - 1$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

$=$ RHS

(ii) Hence solve $16x^4 - 16x^2 + 1 = 0$

let $x = \cos\theta$

$16\cos^4\theta - 16\cos^2\theta + 1 = 0$

$8\cos^4\theta - 8\cos^2\theta + \frac{1}{2} = 0$

$8\cos^4\theta - 8\cos^2\theta + 1 = -\frac{1}{2} + 1$ (1)

$\therefore \cos 4\theta = \frac{1}{2}$

Examination:

Level: Ext 2

Year: 2019

pg 4

QUESTION: 14 cont	Markers Comments

$\therefore 4\theta = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$ (4 roots as it's a quartic)

$\therefore \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{11\pi}{12}$ $\quad \therefore x = \cos\frac{\pi}{12}, \cos\frac{5\pi}{12}, \cos\frac{7\pi}{12}, \cos\frac{11\pi}{12}$

(iii) root 2 at a time $= c/a = -1$

$\cos\frac{\pi}{12}.\cos\frac{5\pi}{12} + \cos\frac{\pi}{12}\cos\frac{7\pi}{12} + \cos\frac{\pi}{12}\cos\frac{11\pi}{12} + \cos\frac{5\pi}{12}\cos\frac{7\pi}{12} + \cos\frac{5\pi}{12}\cos\frac{11\pi}{12} + \cos\frac{7\pi}{12}\cos\frac{11\pi}{12} = -1$

(now $\cos\frac{11\pi}{12} = -\cos\frac{\pi}{12}$ and $\cos\frac{7\pi}{12} = -\cos\frac{5\pi}{12}$)

$\therefore -1 = \cos\frac{\pi}{12}.\cos\frac{5\pi}{12} - \cos\frac{\pi}{12}\cos\frac{5\pi}{12} - \cos\pi/_{12}\cos\pi/_{12} - \cos\frac{5\pi}{12}\cos\frac{5\pi}{12} - \cos\frac{5\pi}{12}\cos\frac{\pi}{12} + \cos\frac{5\pi}{12}\cos\frac{\pi}{12}$

$-1 = -\cos^2\frac{\pi}{12} - \cos^2\frac{5\pi}{12}$

$\therefore \cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} = 1 \quad \ldots$ (a)

product of roots $= \frac{1}{16}$

$\cos\frac{\pi}{12}\cos\frac{5\pi}{12}\cos\frac{7\pi}{12}\cos\frac{11\pi}{12} = \frac{1}{16}$

$\cos^2\frac{\pi}{12}.\cos^2\frac{5\pi}{12} = \frac{1}{16}$ (as $\cos\frac{7\pi}{12} = -\cos\frac{5\pi}{12}$ etc)

$\cos\pi/_{12}\cos\frac{5\pi}{12} = 1/4$ (as in 1st quadrant)

now $\left(\cos\frac{\pi}{12} + \cos\frac{5\pi}{12}\right)^2 = \cos^2\frac{\pi}{12} + \cos^2\frac{5\pi}{12} + 2\cos\frac{\pi}{12}\cos\frac{5\pi}{12}$

$\left(\cos\frac{\pi}{12} + \cos\frac{5\pi}{12}\right)^2 = 1 + 2\times(1/4)$

$\left(\cos\pi/_{12} + \cos\frac{5\pi}{12}\right)^2 = 3/2$

$\cos\frac{\pi}{12} + \cos\frac{5\pi}{12} = \sqrt{3/2}$ (as both angles are in the first quadrant)

*needed to find sum of roots 2 at a time and product of roots to get 1 mark

* 2nd mark for using expansion and supplementary angles

Examination:

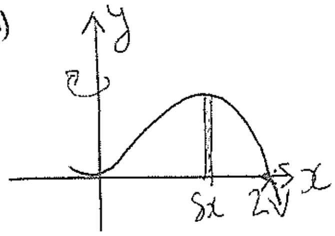
Level: Ext 2

Year: 2019

pg 1

QUESTION: 15	Markers Comments

(a)



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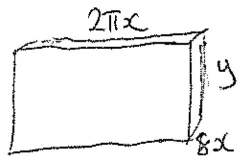


= 1

$\delta V = 2\pi x y\,\delta x$

$= 2\pi x(4x^2 - x^4)\delta x$

$= 2\pi(4x^3 - x^5)\delta x$ 1

$\therefore V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi(4x^3 - x^5)\,\delta x$ 1

$= 2\pi \int_0^2 (4x^3 - x^5)\,dx$

$= 2\pi \left[x^4 - \frac{x^6}{6}\right]_0^2$

$= 2\pi\left(2^4 - \frac{64}{6} - 0 + 0\right)$

$= \frac{32\pi}{3}$ units^3 1

(b)

weight $= mg = 5 \times 10 = 50N$

$F_{net} = 50N - 6N - 0.25N$

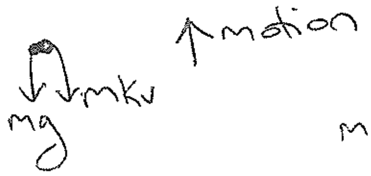
$= 43.75N$

now $F = m\ddot{x}$

$43.75 = m\ddot{x}$

$\ddot{x} = \frac{43.75}{5} = 8.75\ m/s^2$ down. 1

(c) (i)



$F = m\ddot{x}$

$m\ddot{x} = -mg - mkv$

$\ddot{x} = -g - kv$

1 needed to show all the working out to gain the 1 mark.

Examination:

Level: Ext 2

Year: 2019

pg 2

QUESTION: 15	Markers Comments

(c)(ii) $\ddot{x} = \frac{dv}{dt} = -g - kv$

$$\frac{dt}{dv} = \frac{-1}{g+kv}$$

$$dt = \frac{-dv}{g+kv} \quad \text{at } t=0, v=u \quad \text{at } t=T, v=0$$

$$\int_0^T dt = \int_u^0 \frac{-1}{g+kv}\,dv$$ 1

$$\therefore T = \left[-\frac{1}{k}\ln(g+kv)\right]_u^0$$ 1

$$= -\frac{1}{k}\left[\ln(g+0) - \ln(g+ku)\right]$$

$$= \frac{1}{k}\left[\ln(g+ku) - \ln(g)\right]$$

$$= \frac{1}{k}\left[\ln\left(\frac{g+ku}{g}\right)\right] \text{ seconds.}$$ 1

(iii) $\ddot{x} = -g - kv$

$$v\frac{dv}{dx} = -g - kv$$

$$\frac{dv}{dx} = \frac{-g-kv}{v}$$

$$\therefore \frac{dx}{dv} = \frac{-v}{g+kv}$$ 1

At $t=0, x=0, v=u$

at $x=H, v=0$ (max height)

$$\int_0^H dx = \int_u^0 \frac{-v}{g+kv}\,dv$$

$$H = -\frac{1}{k}\int_u^0 \frac{kv}{g+kv}\,dv$$

Examination:

Level: Ext 2

Year: 2019

QUESTION:	Markers Comments

$$H = -\frac{1}{k}\int_u^0 \left(\frac{kv+g}{kv+g} - \frac{g}{g+kv}\right)dv$$

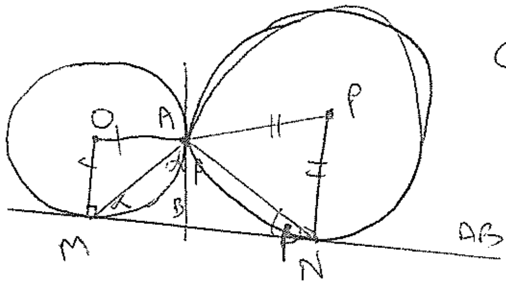
$$= -\frac{1}{k}\int_u^0 1\,dv + \frac{g}{k^2}\int_u^0 \frac{k}{g+kv}\,dv \qquad (1)$$

$$H = -\frac{1}{k}\left[v\right]_u^0 + \frac{g}{k^2}\left[\ln(g+kv)\right]_u^0$$

$$= \frac{u}{k} + \frac{g}{k^2}\ln g - \frac{g}{k^2}\ln(g+ku)$$

$$\therefore H = \frac{u}{k} + \frac{g}{k^2}\ln\left(\frac{g}{g+ku}\right) \text{ metres} \qquad (1)$$

(d)



Circles are centred at O and P respectively.

AB is common tangent.

*Poorly attempted.
*Many ways to do this but need to clearly label 2 angles as α and β. State full reasons

(1) let $\angle AMN = \alpha$ and $\angle ANM = \beta$

$\angle OMN = 90°$ (tangent is perpendicular to radius at the point of contact).

similarly $\angle PNM = 90°$

$\therefore \angle OMA = 90° - \alpha$ (subtraction of adjacent angles)

$OM = ON$ (both radii)

$\angle OAM = \angle OMA$ (equal angles are opposite equal sides in $\triangle OAM$)

$= 90 - \alpha$

(1) $\angle OAB = 90°$ (tangent AB perpendicular to radius at pt. of contact)

$\therefore \angle MAB = 90° - (90° - \alpha)$ (subtraction of adjacent angles)

$= \alpha$

Similarly $\angle NAB = \beta$

$\therefore$ In $\triangle MAN$, $\alpha + \beta + \alpha + \beta = 180°$ (angle sum of $\triangle MAN$)

$2\alpha + 2\beta = 180°$

$\alpha + \beta = 90°$

(1) $\therefore \angle MAN = \alpha + \beta = 90°$ (sum of adjacent angles)

3/7

Question 16

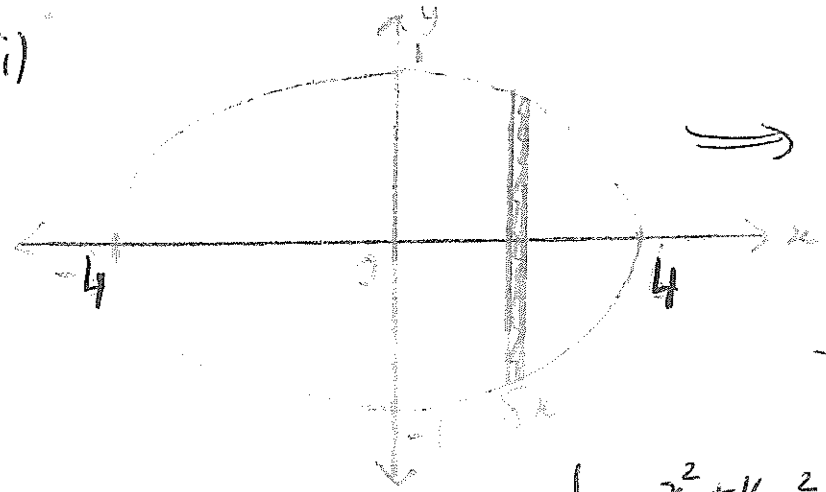
a) i) $x^2 + 16y^2 = 16$

$$\frac{x^2}{16} + \frac{y^2}{1} = 1 \quad , \text{hence } a = 4 \ , b = 1 .$$

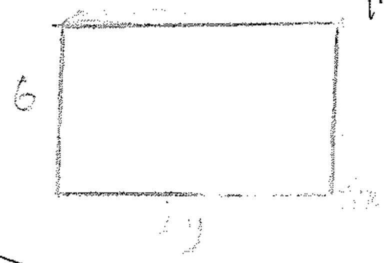
$$\text{Area} = \pi ab = \pi(4)(1) = 4\pi \text{ units}^2$$

many tried by using integration technique rather than using the formula

ii)



α)

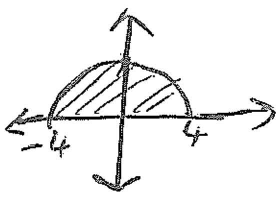


good attempt

β)
$$\delta V = (6 \times 2y)\,\delta x$$
$$= 12y\,\delta x$$
$$= 12 \times \frac{\sqrt{16-x^2}}{4}\,\delta x$$
$$\therefore \delta V = 3\sqrt{16-x^2}\,\delta x$$

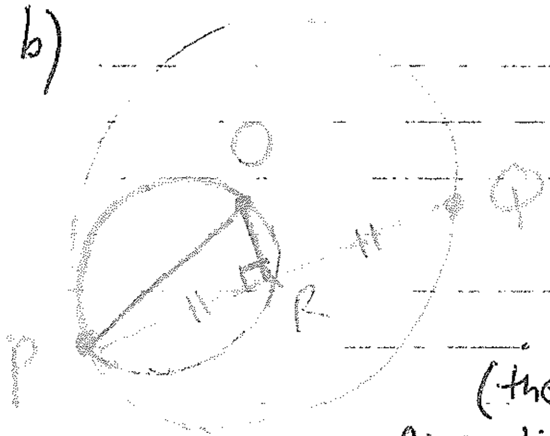
$$x^2 + 16y^2 = 16$$
$$16y^2 = 16 - x^2$$
$$y^2 = 1 - \frac{x^2}{16}$$
$$y = \pm\sqrt{1 - \frac{x^2}{16}}$$
$$\therefore y = \pm\frac{\sqrt{16-x^2}}{4}$$

γ)
$$V = \lim_{\delta x \to 0} \sum_{x=-6}^{x=6} 3\sqrt{16-x^2}\,\delta x$$
$$= 3\int_{-6}^{6} \sqrt{16-x^2}\,dx$$
$$= 3 \times \frac{1}{2} \times \pi \times 4^2$$
$$\therefore V = 24\pi \text{ units}^3$$



Area of semi-circle radius 4 units

b)



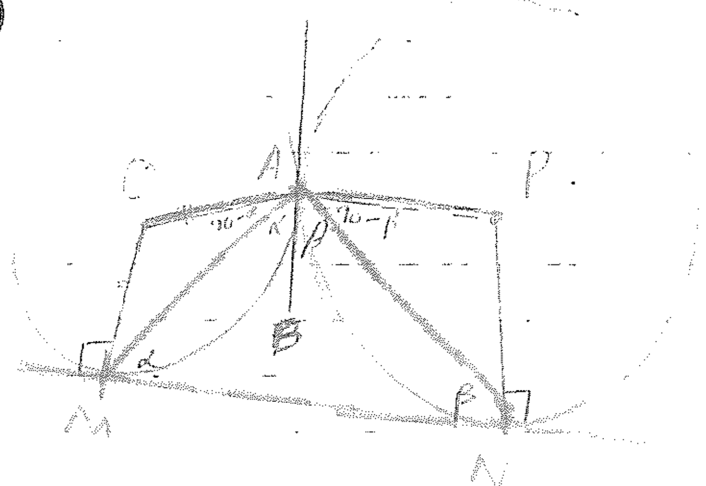
Not many attempt this question

$\angle ORP = 90°$
(angle in a semicircle, circle with diameter OP)

$\therefore PR = RQ$
(the line through the centre of circle centre perpendicular to chord bisects that chord)

$\therefore$ R is always the midpoint of PQ

15 d)

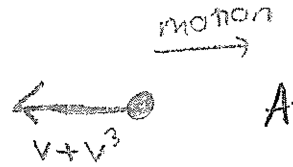


Circles have centre O and P respectively

Let $\angle AMN = \alpha$ and $\angle ANM = \beta$
$\angle OMN = \angle PNM = 90°$ (tangent is perpendicular to radius at the point of contact)

$\angle OMA = 90° - \alpha$
Since $OM = OA$ (radii of circle, centre O)
$\angle OAM = 90° - \alpha$ ($\angle$'s opposite equal sides in $\triangle OAM$)
$\angle OAB = 90°$ (tangent $\perp$ radius at the point of contact, circle O)
$\therefore \angle MAB = 90° - (90° - \alpha)$
$= \alpha$
Similarly, $\angle NAB = \beta$

In $\triangle MAN$, $\alpha + \beta + \alpha + \beta = 180°$ (angle sum of $\triangle MAN$)
$\therefore \alpha + \beta = 90°$
$\therefore \angle MAN = \alpha + \beta = 90°$ (adjacent angles)

Q4) 

At $t=0, x=0$ and $v=K$

i) $F = m\ddot{x} = -(v+v^3)$

$\therefore \ddot{x} = -(v+v^3)$ as $m = 1\,kg$

$$\ddot{x} = v\frac{dv}{dx} = -(v+v^3)$$

$$\frac{dv}{dx} = -(1+v^2)$$

$$\frac{dx}{dv} = \frac{-1}{1+v^2}$$

$$\int dx = \int \frac{-1}{1+v^2}\,dv$$

$$x = -\tan^{-1}(v) + C$$

At $t=0$, $x=0$, $v=K$:

$$0 = -\tan^{-1}(K) + C$$

$$\therefore C = \tan^{-1}(K)$$

$$\therefore x = \tan^{-1}(k) - \tan^{-1}(v)$$

Many reached up to this line, but struggled using double angle formula.

$$\tan x = \tan\left[\tan^{-1}(K) - \tan^{-1}(v)\right]$$

$$= \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$= \frac{K-v}{1+kv}$$

Let $\alpha = \tan^{-1}k$
$\beta = \tan^{-1}v$
$\Downarrow$
$\tan\alpha = k$
$\tan\beta = v$

$$\therefore x = \tan^{-1}\left(\frac{K-v}{1+kv}\right)$$

ii) $$\ddot{x} = \frac{dv}{dt} = -(v+v^3)$$

$$\frac{dt}{dv} = \frac{-1}{v+v^3} = \frac{-1}{v(1+v^2)}$$

$$\frac{-1}{v(1+v^2)} \equiv \frac{A}{v} + \frac{Bv+C}{1+v^2}$$ where A, B, C are constants

$$-1 \equiv A(1+v^2) + v(Bv+C)$$

Sub $v=0$:

$$-1 = A(1+0) + 0$$

$$\therefore \underline{\underline{A=-1}}$$

Sub $v=i$: $-1 = A(1+i^2) + i(Bi+C)$

$$-1 = Bi^2 + Ci$$

$$-1 = Ci - B$$

Equating real parts $\Rightarrow$ $\underline{\underline{B=1}}$

Equating imaginary parts $\Rightarrow$ $\underline{\underline{C=0}}$

$$\therefore \frac{dt}{dv} = \frac{-1}{v} + \frac{v}{1+v^2}$$

many struggled with separating it but somehow reached at the answer !!

$$\int_0^T dt = \int_K^v \frac{-1}{v} + \frac{v}{1+v^2}\, dv$$

$$T = \left[-\ln(v) + \tfrac{1}{2}\ln(1+v^2)\right]_K^v$$

$$= -\ln(v) + \tfrac{1}{2}\ln(1+v^2) + \ln(K) - \tfrac{1}{2}\ln(1+k^2)$$

$$= \ln\left(\frac{K}{v}\right) + \frac{1}{2}\ln\left(\frac{1+v^2}{1+k^2}\right)$$

$$= \frac{1}{2}\ln\left(\frac{K^2}{v^2}\right) + \frac{1}{2}\ln\left(\frac{1+v^2}{1+k^2}\right)$$

$$\therefore T = \frac{1}{2}\ln\left(\frac{K^2(1+V^2)}{V^2(1+k^2)}\right)$$ seconds